Chapter 3 Table of Arithmetic Functions

Arithmetic fucntion seen in Chapter 3, their values on prime powers and their factorizations.

Arithmetic Function f	$f(p^r)$	
Divisor function $d(n) = \sum_{d\mid n} 1 = \sum_{ab=n} 1$	$d(p^r) = r{+}1$	$d = 1 * 1$
$d_k(n) = \sum_{a_1 a_2 \ldots a_k = n} 1$	$d_k(p^r) = \binom{r+k-1}{k-1}$	$d_k = \underbrace{1 * 1 * \ldots * 1}_{k \text{ times}}$
$\sigma(n) = \sum_{d\mid n} d$	$\sigma(p^r) = \dfrac{p^{r+1}-1}{p-1}$	$\sigma = 1 * j$
$\sigma_\nu(n) = \sum_{d\mid n} d^\nu;\ \nu \in \mathbb{C}$	$\sigma_\nu(p^r) = \dfrac{p^{\nu(r+1)}-1}{p^\nu - 1}$	$\sigma_\nu = 1 * j^\nu$
Euler's totient function $\phi(n) = \sum_{\substack{1\le r\le n \\ \gcd(r,n)=1}} 1$	$\phi(p^r) = p^{r-1}(p{-}1)$ for $r \geq 1$,	$\phi = \mu * j$
$\omega(n) = \sum_{p\mid n} 1$	$\omega(p^r) = 1$	
$\Omega(n) = \sum_{p^a\mid\mid n} a = \sum_{p\mid n} \sum_{\substack{r\ge 1 \\ p^r\mid n}} 1$	$\Omega(p^r) = r.$	
$1(n) = 1$	$1(p^r) = 1$	
$j(n) = n$	$j(p^r) = p^r$	
$\delta(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$	$\delta(p^r) = 0$ if $r \geq 1$,	$\delta = 1 * \mu$
Möbius function For $n = p_1^{a_1} p_2^{a_2} \ldots p_r^{a_r}$ $\mu(n) = \begin{cases} (-1)^r & \text{if } \forall i,\ a_i = 1 \\ 0 & \text{if } \exists i : a_i \geq 2. \end{cases}$	$\mu(p^r) = \begin{cases} -1 & \text{if } r = 1 \\ 0 & \text{if } r \geq 2. \end{cases}$	

Arithmetic Function f	$f(p^r)$	
Liouville function $\lambda(n) = (-1)^{\Omega(n)}$	$\lambda(p^r) = (-1)^r$,	$\lambda * Q_2 = \delta$
Square-full numbers $q_2(n) = \begin{cases} 1 & \text{if } p\|n \Rightarrow p^2\|n, \\ 0 & \text{otherwise.} \end{cases}$	$q_2(p^r) = \begin{cases} 1 & \text{if } r \geq 2, \\ 0 & \text{if } r = 1. \end{cases}$	
Square-free numbers $Q_2(n) = \begin{cases} 0 & \text{if } \exists p : p^2\|n, \\ 1 & \text{otherwise.} \end{cases}$	$Q_2(p^r) = \begin{cases} 1 & \text{if } r = 1, \\ 0 & \text{if } r \geq 2. \end{cases}$	$Q_2(n) = \|\mu(n)\| = \mu^2(n)$
$\mu_2(n) = \begin{cases} \mu(m) & \text{if } n = m^2, \\ 0 & \text{otherwise.} \end{cases}$	$\mu_2(p^r) = \begin{cases} -1 & \text{if } r = 2, \\ 0 & \text{otherwise.} \end{cases}$	$\mu_2 = \mu * Q_2$ so $Q_2 = 1 * \mu_2$
k-full numbers $q_k(n) = \begin{cases} 1 & \text{if } p\|n \Rightarrow p^k\|n \\ 0 & \text{otherwise} \end{cases}$	$q_k(p^r) = \begin{cases} 1 & \text{if } r \geq k \\ 0 & \text{if } r \leq k-1. \end{cases}$	
k-free numbers $Q_k(n) = \begin{cases} 0 & \text{if } \exists p : p^k\|n, \\ 1 & \text{otherwise.} \end{cases}$	$Q_k(p^r) = \begin{cases} 1 & \text{if } r \leq k-1, \\ 0 & \text{if } r \geq k. \end{cases}$	
$\mu_k(n) = \begin{cases} \mu(m) & \text{if } n = m^k, \\ 0 & \text{otherwise.} \end{cases}$	$\mu_k(p^r) = \begin{cases} -1 & \text{if } r = k, \\ 0 & \text{otherwise.} \end{cases}$	$\mu_k = \mu * Q_k$, so $Q_k = 1 * \mu_k$.
$2^{\omega(n)}$	$2^{\omega(p^r)} = 2$	$\begin{cases} 1 * 1 * \mu_2 \\ 1 * Q_2 \end{cases}$
$2^{\Omega(n)}$	$2^{\Omega(p^r)} = 2^r$	
$sq(n) = \begin{cases} 1 & \text{if } n = m^2, \\ 0 & \text{otherwise.} \end{cases}$	$sq(p^r) = \begin{cases} 1 & \text{if } 2\|r, \\ 0 & \text{otherwise.} \end{cases}$	$sq = 1 * \lambda$ $sq * \mu_2 = \delta$
$g(n) = d(n^2)$	$g(p^r) = d(p^{2r}) = 2r+1$	$\begin{cases} 1 * 1 * 1 * \mu_2 \\ 1 * 1 * Q_2 \\ 1 * 2^{\omega} \end{cases}$
$d^2(n)$	$d^2(p^r) = (r+1)^2$	$\begin{cases} 1 * 1 * 1 * 1 * \mu_2 \\ 1 * 1 * 1 * Q_2 \\ 1 * 1 * 2^{\omega} \\ 1 * g \end{cases}$

Arithmetic Function f	**Dirichlet Series** $D_f\,(s)$
Divisor function $d(n) = \sum_{d\mid n} 1 = \sum_{ab=n} 1$	$\zeta^2(s)$
$d_k(n) = \sum_{a_1 a_2 \ldots a_k = n} 1$	$\zeta^k(s)$
$\sigma(n) = \sum_{d\mid n} d$	$\zeta(s)\,\zeta(s-1)$ for $\mathrm{Re}\, s > 2$
$\sigma_\nu(n) = \sum_{d\mid n} d^\nu;\ \nu \in \mathbb{C}$	$\zeta(s)\,\zeta(s-\nu)$ for $\mathrm{Re}\,(s-\nu) > 1$
Euler's totient function $\phi(n) = \sum_{\substack{1\le r\le n \\ \gcd(r,n)=1}} 1$	$\frac{\zeta(s-1)}{\zeta(s)}$
$1(n) = 1$	$\zeta(s)$
$j(n) = n$	$\zeta(s-1)$
$\delta(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$	1
Möbius function $\mu(n) = (-1)^{\omega(n)}$	$\frac{1}{\zeta(s)}$

Arithmetic Function f	**Dirichlet Series** $D_f(s)$
Liouville function $\lambda(n) = (-1)^{\Omega(n)}$	$\frac{\zeta(2s)}{\zeta(s)}$
Square-full numbers $q_2(n) = \begin{cases} 1 & \text{if } p\|n \Rightarrow p^2\|n, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{\zeta(2s)\zeta(3s)}{\zeta(6s)}$
Square-free numbers $Q_2(n) = \begin{cases} 0 & \text{if } \exists p : p^2\|n, \\ 1 & \text{otherwise.} \end{cases}$	$\frac{\zeta(s)}{\zeta(2s)}$
$\mu_2(n) = \begin{cases} \mu(m) & \text{if } n = m^2, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{1}{\zeta(2s)}$ for $\operatorname{Re} s > 1/2$.
k-free numbers $Q_k(n) = \begin{cases} 0 & \text{if } \exists p : p^k\|n, \\ 1 & \text{otherwise.} \end{cases}$	$\frac{\zeta(s)}{\zeta(ks)}$ for $\operatorname{Re} s > 1/k$
$\mu_k(n) = \begin{cases} \mu(m) & \text{if } n = m^k, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{1}{\zeta(ks)}$ for $\operatorname{Re} s > 1/k$
$2^{\omega(n)}$	$\frac{\zeta^2(s)}{\zeta(2s)}$
$sq(n) = \begin{cases} 1 & \text{if } n = m^2, \\ 0 & \text{otherwise.} \end{cases}$	$\zeta(2s)$
$g(n) = d(n^2)$	$\frac{\zeta^3(s)}{\zeta(2s)}$
$d^2(n)$	$\frac{\zeta^4(s)}{\zeta(2s)}$